

EXACT SUM RULES AT FINITE TEMPERATURE AND CHEMICAL POTENTIAL AND THEIR APPLICATION TO QCD*

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ABSTRACT

Within the framework of the operator product expansion (OPE) and the renormalization group equation (RGE), we show that the temperature and chemical potential dependence of the zeroth moment of a spectral function (SF) in an asymptotically free theory is completely determined by the one-loop structure of the theory. This exact result constrains the qualitative shape of SF's, and implies striking phenomenological effects near phase transitions.

Our present understanding of QCD at finite temperature (T) and baryon density (or chemical potential μ) is mainly limited in the Euclidean realm, due to the lack of non-perturbative and systematic calculating tools directly in the Minkowski space. The typical methods, with QCD Lagrangian as the starting point, are the OPE and lattice simulation. Because of these two formulations are intrinsically Euclidean, only static quantities are conveniently studied. In order to gain dynamical informations, which are more accessible experimentally, the analytic structure implemented through dispersion relations often have to be invoked within the theory of linear response.

The real-time linear response to an external source coupled to a renormalized current $J(x)$ is given by the retarded correlator:

$$K(x; T, \mu) \equiv \theta(x_0) \langle [J(x), J(0)] \rangle_{T, \mu}, \quad (1)$$

where the average is on the grand canonical ensemble specified by (T, μ) . Disregarding possible subtraction terms, the following dispersion relation for the frequency dependence of the retarded correlator can be written:

$$\tilde{K}(\omega, \mathbf{k}; T, \mu) = \int_0^\infty du^2 \frac{\rho(u, \mathbf{k}; T, \mu)}{u^2 - (\omega + i\epsilon)^2}. \quad (2)$$

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For convenience, we discuss only the uniform limit ($\mathbf{k} = 0$). Upon analytic continuation, $\omega \rightarrow iQ$, the dispersion relation in principle connects the correlator in Euclidean region to the SF, which embodies all the real-time information. However, given only approximate knowledge of $\tilde{K}(iQ; T, \mu)$, either via the OPE or lattice calculations, it is extremely difficult technically to “invert” the dispersion relation. Only when we have enough understanding of the SF involved this “inverting” can be made practical. In fact, the success of the so-called QCD sum rule approach and analyzing lattice data under many situations rests, to some extent, fortuitously on the successful parameterization of SF’s.

On the other hand, the lack of adequate understanding of SF’s at finite (T, μ) severely limits our chances to extract physical information using these approaches. It is likely that physical results at finite (T, μ) can be strongly biased, if one naively assumes the same parameterizations of the SF’s that work at zero (T, μ) . The purpose of this work is to derive exact sum rules that constrain the variation of SF’s with (T, μ) . The derivation, based on the OPE and the RGE, has close analogies to the analysis of deep inelastic lepton scattering experiments. In addition, we apply these sum rules to the chiral phase transition, and demonstrate that SF’s in some channels are drastically modified compared both to their zero (T, μ) and perturbative shape.

In an asymptotically free theory, the OPE yields, *e.g.* in the $\overline{\text{MS}}$ scheme, the large- Q^2 asymptotic expansion

$$\tilde{K}(iQ; T, \mu, \kappa) \sim \tilde{K}_0(iQ, \kappa) + \sum_n C_n(Q^2, g^2(\kappa), \kappa) \langle [O_n]_\kappa \rangle_{T, \mu}, \quad (3)$$

where $g^2(\kappa)$, $[O_n]_\kappa$ ’s and C_n ’s are, respectively, the coupling constant, the renormalized composite operators and their corresponding Wilson coefficients at the subtraction mass scale κ . It is important to notice that the information of the ensemble average is encoded in the matrix elements of the composite operators, while the Wilson coefficients and \tilde{K}_0 are independent of T and μ . Although the matrix elements $\langle [O_n]_\kappa \rangle_{T, \mu}$ cannot be determined perturbatively, the Q^2 -dependence of Wilson coefficients C_n is dictated by the renormalization group equation and given by

$$C_n(Q^2, g(\kappa), \kappa) = \frac{c_n(g^2(Q))}{Q^{d_n}} \left[\frac{g^2(Q)}{g^2(\kappa)} \right]^{(2\gamma_J - \gamma_n)/2b} \left\{ 1 + \mathcal{O}(g^2(Q)) \right\}, \quad (4)$$

where d_n is the canonical dimension of the operator O_n minus the dimension of \tilde{K} and $c_n(g^2(Q))$ is calculable perturbatively. The pure numbers γ_i ($i = J, n$) and b are related to the anomalous dimensions of J , O_n and to the β -function as follows

$$\Gamma_i = -\gamma_i g^2 + \mathcal{O}(g^4), \quad \text{and} \quad \beta = -b g^4 + \mathcal{O}(g^6). \quad (5)$$

To study the dependence of \tilde{K} on (T, μ) we only need to consider the difference $\Delta\tilde{K}(iQ) \equiv \tilde{K}(iQ; T, \mu) - \tilde{K}(iQ; T', \mu')$ and

$$\Delta\tilde{K}(iQ) = \int_0^\infty du^2 \frac{\Delta\rho(u)}{u^2 + Q^2}, \quad (6)$$

where $\Delta\rho(u) \equiv \rho(u; T, \mu) - \rho(u; T', \mu')$. This subtraction is crucial to remove $\tilde{K}_0(iQ, \kappa)$, which contains all the terms not suppressed by a power of $1/Q^2$, and also to make $\Delta\tilde{K}(iQ)$ independent of the renormalization point κ . Finite masses give corrections of order $m^2(Q)/Q^2$, with $m^2(Q)$ that runs logarithmically and hence can be ignored, if we are only interested in the lowest moment of the subtracted SF.

At this point we have expressed the left-hand side of Eq. (6) as an asymptotic expansion of the form:

$$\Delta\tilde{K}(iQ) \sim \sum_{n,\nu=0}^{\infty} \frac{c_n^{(\nu)}(\kappa)\Delta\langle[O_n]_{\kappa}\rangle}{Q^{d_n}} [g^2(Q)]^{\nu+\eta_n}, \quad (7)$$

where $\Delta\langle[O_n]_{\kappa}\rangle$ denotes the difference between the expectation values of $[O_n]_{\kappa}$ in the ensembles specified by (T, μ) and (T', μ') and the exponent η_n and the Q^2 -independent coefficients $c_n^{(\nu)}(\kappa)$ are known perturbatively.

We proceed by making an analogous asymptotic expansion of $\Delta\rho(u)$:

$$\Delta\rho(u) \sim \sum_{n=0}^{\infty} \frac{[g^2(u)]^{\xi_n}}{u^{2(n+1)}} \sum_{\nu=0}^{\infty} a_n^{(\nu)} [g^2(u)]^{\nu}. \quad (8)$$

For notational clarity, we have ignored exponentially suppressed terms and the fact that there can be more than one η_n and ξ_n for each n . We then obtain the sum rules by imposing that the asymptotic expansion of the right-hand side of Eq. (6), which we get by inserting Eq. (8) in the dispersion integral, matches the left-hand side obtained by the OPE. In a long paper¹ we shall present those technical details that make the procedure sketched above rigorous. At the moment we only need to consider the leading terms in the expansions of, respectively, the left-hand side and the right-hand side of Eq. (6):

$$[g^2(Q)]^{\eta_n} \frac{\Delta\langle[O_n]_{\kappa}\rangle}{Q^{d_n}} [c_n^{(0)}(\kappa) + c_n^{(1)}(\kappa)g^2(Q)], \quad (9)$$

$$\frac{1}{Q^2} \left[\overline{\Delta\rho} + \frac{a_0^{(0)}}{(1-\xi_n)} [g^2(Q)]^{\xi_n-1} \right]. \quad (10)$$

In Eq. (10), if $\xi_n > 1$, $\overline{\Delta\rho}$ can be shown to be equal to the zeroth moment of the subtracted SF; note that the zeroth moment of a function whose asymptotic expansion is Eq. (8) is infinite, if $\xi_n \leq 1$, since $\int_A^\infty dx x^{-1} (\log x)^{-\xi_n} = \infty$.

For the sake of concreteness, let us examine the consequences of matching Eq. (9) and Eq. (10) in the case we are concerned with, *i.e.* $n = 1$ and $d_1 = 2$. If the OPE calculation produces $\eta_n = 0$, then ξ_n must be an integer greater than one, the zeroth moment exists and is given by

$$\int_0^\infty du^2 \Delta\rho(u) = c_1^{(0)}(\kappa) \Delta\langle[O_1]_{\kappa}\rangle. \quad (11)$$

If $\eta_n > 0$, then $\xi_n = 1 + \eta_n$, the zeroth moment is again finite and equal to zero:

$$\int_0^\infty du^2 \Delta\rho(u) = 0. \quad (12)$$

Our main results, Eq. (11) and Eq. (12), can be expressed in physical terms as follows. The zeroth moment of a SF for a current J whose OPE expansion yields $\eta_n > 0$ is independent of T and μ , while the same moment for a current with $\eta_n = 0$ changes with T and μ proportionally to the corresponding change(s) of the condensate(s) of leading operator(s). When $\xi_n < 1$ or $\xi_n = 1$ in Eq. (8) terms such as $[g^2(Q)]^{-1}$ or $\ln[g^2(Q)]$ in Eq. (10) would be produced. Hence the appearance of the inverse power of the $g^2(Q)$ or $\ln[g^2(Q)]$ in the OPE series is a indication that the zeroth moment of the subtracted SF is infinite.

At this point several more general comments are appropriate: 1) It is essential to take into account the QCD logarithmic corrections, for the logarithmic corrections not only dictate whether $\overline{\Delta\rho}$ satisfy Eq.(11) or Eq.(12), but also control the very existence of $\overline{\Delta\rho}$. 2) The derivation of sum rules for higher moments of the SF requires the complete cancelation of all the lower dimensional operator terms, not just the leading $g^2(Q)$ terms; in particular, we also need current quark mass corrections to the Wilson coefficients. An appropriate subtraction is the prerequisite for the convergence of higher moments. 3) The (T, μ) -dependent part of the leading condensate appearing in Eq. (11) does not suffer from the infrared renormalon ambiguity, because only the perturbative term \tilde{K}_0 can generate contributions to the leading condensate that are dependent on the prescription used to regularize these renormalons. But \tilde{K}_0 is independent of T and μ and any prescription dependence cancels out when we make the subtraction in Eq. (6). On the contrary, unless we generalize Eq. (6) and make other subtractions, sum rules that involve non-leading condensates are, in principle, ambiguous. 4) We have explicitly verified the correctness of our results in a soluble model,¹ the Gross-Neveu model in the large- N limit, where we obtain exactly all the relevant quantities, such as SF's at arbitrary (T, μ) , Wilson coefficients, β - and Γ -functions in pseudoscalar and vector channels.

Now let us specialize to QCD and consider four correlation functions: two involving the non-conserved scalar $J_S = \bar{\psi}\psi$ and pseudoscalar $J_P = \bar{\psi}\gamma_5\psi$ currents ($\gamma_{J_S} = \gamma_{J_P} = 1/4\pi^2$) and two involving the conserved vector $J_V = \bar{\psi}\gamma_\mu\psi$ and axial-vector $J_A = \bar{\psi}\gamma_\mu\gamma_5\psi$ currents ($\gamma_V = \gamma_A = 0$). Since the leading operators (dimension four) have non-positive anomalous dimensions, the two non-conserved currents have $\eta_n \geq (2\gamma_J - \gamma_n)/2b > 0$ and Eq. (12) applies, *i.e.* the zeroth moments of their SF's are independent of T and μ . On the other hand, the two conserved currents have $\eta_n = 0$ and a generalization of Eq. (11) applies.¹ Since there are three dimension-four operators with zero anomalous dimension, the sum rules for the vector and axial-vector currents are

$$\int_0^\infty du^2 \Delta\rho(u) = a\Delta\langle[m\bar{\psi}\psi]\rangle + \frac{\Delta\langle[\alpha_s G^2]\rangle}{2\pi} + 8\Delta\langle[\theta_{00}]\rangle, \quad (13)$$

where $\theta_{\mu\nu}$ is the traceless stress tensor and $a = 6$ for vector and $a = -10$ for axial channels respectively. This exact sum rule should not be contaminated explicitly by instantons, although the value of the condensates certainly have instanton contributions. The reason is that the instanton singularities in the Borel-plane are located on the positive axis starting at $8\pi^2$, and, therefore, contribute to correlation functions only with higher order terms of $1/Q^2$ in the OPE series.

Finally, let us discuss some of the phenomenological consequences of these exact

sum rules. In the pseudoscalar channel, $\overline{\Delta\rho} = 0$ implies that, in the broken-chiral-symmetry phase, the change of the pion pole induced by T or μ is exactly compensated by a corresponding change of the continuum part of the SF. Next let us consider the scalar correlation function at $Q^2 = 0$, the chiral susceptibility,

$$\chi(T, \mu) \equiv \int d^4x \theta(x_0) \langle [J_S(x), J_S(0)] \rangle_{T, \mu} = \int_0^\infty du^2 \frac{\rho(u; T, \mu)}{u^2}, \quad (14)$$

which diverges when (T, μ) approaches the phase boundary, provided the chiral restoration is a continuous transition. The divergence of the chiral susceptibility near phase transition only can be produced in Eq.(14) by singularities very close to the origin, when the exact sum rule $\overline{\Delta\rho} = 0$ is simultaneously taken into account. Thus the spectral function would have to have a vanishing threshold (since there is no massless poles in the chirally symmetric phase) and develop a strong peak right above the threshold, when (T, μ) is very close to the phase boundary. Because in the chirally symmetric phase the pseudoscalar and scalar channels are degenerate, the same would also happen to the pseudoscalar SF. This strong peak in the pseudoscalar and scalar SF's, which is intimately connected with the critical phenomenon of diverging susceptibility and the correlation length near the phase transition, can be interpreted as some kind of quasi-particle, thus confirming the qualitative picture, originally proposed in the context of the Nambu-Jona-Lasinio model,² of the appearance of soft modes near the chiral phase transition. Similar situation, though less drastic, can be argued to occur in vector and axial channels, based on the facts that $\Delta\langle[\theta_{00}]\rangle$, $\Delta\langle[m\bar{\psi}\psi]\rangle$ and $\Delta\langle[\alpha_s G^2]\rangle$ behave smoothly across the critical line and the rapid increase of the so-called baryon number susceptibility at the chiral restoration point in lattice simulations.³ If the chiral restoration turns out not to be a second order phase transition, but rather a cross-over or weak first order transition (finite but large correlation length), as the lattice data seem to indicate,⁴ we expect the same qualitative features, though less pronounced.

In summary, we used OPE and RGE to derive exact sum rules at finite (T, μ) in asymptotically free theories. These exact sum rules strongly constrain the qualitative shape of SF's, especially near phase transition region. We urge whoever parameterizes a SF, *e.g.* in the QCD sum rule calculations or to interpret lattice simulations, to incorporate these exact constraints. In the future, we plan to generalize these results to baryonic currents and analyze their phenomenological consequences in greater detail.

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